"Quantum electrodynamics" – The technical part of the lecture

M: Now we're ready for the technical part of my talk on quantum electrodynamics. First, I'd like to introduce you to one of my colleagues from Theoretical Physics, Dr Formalia. Thanks for coming down today.

S: Thank you for having me here.

Chapter 1

1 a. Spectrum of visible light

M: The spectrum of visible light ranges from red to blue light. The energy of blue light is higher than the energy of red light.

S: The energy is proportional to the frequency ν and comes in packets: Energy = h ν, with h being Planck's constant and ν the frequency.

M: But light is a kind of wave, too, isn't it?

S: Indeed. The wave length λ times frequency ν equals the speed of light c.

M: The wave length of red light is some 800 nanometers and the wave length of blue light is about 400 nanometers.

S: Here we have the two constants of nature c and h, which the entire movie is about, and their relation to light.

1 b. The speed of light in the medium

M: The light seems to bend when passing from water to air – or vice versa, from air into water. Why doesn't it take the straight path? To get to an understanding of the phenomenon, use the following analogy:

Let's assume I was trying to get from the beach to the boat as fast as possible. On the beach, I am able to run faster than in the water.

S: So you'd be foolish if you took the straight path, as it's not the quickest one available.

M: Exactly, for getting to the boat the quickest, you need to cover a longer distance on the beach because the beach allows you to go faster. Eventually, you turn to the sea and then cover the remaining distance wading through the water. The light behaves the same way, as it travels faster in air than it does in water.

Let's see which of these different paths is the fastest. We'll be taking the time needed to cover each of the paths marked in the figure above. Connecting the green points representing the measured times, we get a curve. The curve's minimum indicates the quickest path. This is the path that the light takes.

S: To put it in a more mathematically precise manner: Presupposing the speed in air was c1 and the speed in water was c2, then the law of refraction reading 'sine α divided by sine β equals c1 divided by c2' is equivalent to the statement: "Light takes the fastest path available". The angle β is smaller than the angle α, because light travels at slower speeds in water than it does in air. However, it is not possible for α to exceed 90°. For higher values of α, there is no such solution.

M: In experimental terms that means in the case of α exceeding 90° the light is unable to exit the water.

S: In this case, total reflection of the light occurs.

1 c. Polarization of light

M: Light has two polarization directions. Now what is a polarization direction? Here's a whole lot of light coming towards us. This arrow indicates the direction in which the light moves towards us. Orthogonal to arrow, there's a plane. This plane has two dimensions, # 1 and # 2. The electromagnetic wave's electric field vector is directed orthogonal to the propagation direction of light, i.e. in direction # 1 or direction # 2.
Let's set up a polarization filter right in front of us. Only one of the two directions of propagation may pass through this filter. The other one is filtered out. When inserting a second filter, the light, which after passing through the first filter only swings in the direction # 1, is filtered once again. In the event that the second filter keeps back exactly the light that swings in direction #2, the filter will be dark.

S: In general terms, what we have here are the directions # 1 and # 1' with the angle α in between. The intensity of the light still passing through both filters is proportional to $(\sin \alpha)^2$. Only with the angle α being equal to 90° there will be total darkness. I f the two directions # 1 and # 1' are parallel to each other, the second filter has no effect at all.

M: In modern IMAX theatres, so-called polarizing spectacles are used, which let polarized light pass through on the left eye in the direction # 1 and on the right eye in the direction # 2. Without polarizing spectacles, the 3D movie looks blurred, as two pictures slightly shifted in space appear simultaneously. The specs separate the two pictures, causing a three-dimensional image to be created in the human brain.

1 d. Interference

M: Next we'll deal with a phenomenon called interference. This red light spot comes from the light of a laser pointer. When putting some silk stockings in the way a so-called interference pattern is cast on the wall. What was a single light spot a few seconds ago, is now a conglomeration of many spots. The pattern moves because we stretch and squeeze the silk stockings.

S: Let's take a look at this wave model: The silk stockings have many, many pores. To keep it simple we should examine just two of these. Out of each pore comes a light wave. The crucial point is that the light waves superpose with each other, that is, they interfere. Mathematically, this can be expressed by one wave, (sin x) coming through one pore and sin $(x + \delta x)$ coming through the other pore. We just add them up. By adding up different waves, the interference is created. The phase difference between these waves is δx – and this phase difference is crucial. If δx equals a whole wave length, then both waves add up to (2 sin x).

M: That describes a constructive superposition, with the waves reinforcing one another.

S: If the phase difference amounts to half the wave length, then both waves interfere destructively; because sin x minus sine x is zero.

M: In experimental terms, destructive interference means darkness. Constructive interference, in turn, means brightness – in places with bright spots, the phase difference is a multiple of the wave length.

What happens when we expand the silk stockings in such a way that the distance between pores increases?

S: Well, let's call the distance between pores g, as in grid spacing. If g grows, will the distance d between the bright spots on the wall increase or decrease?

We can compute that using our simple model. We understand that in the bright spots, the phase difference is given by the wave length λ, or 2λ, 3λ, and so on. . On the right-hand side we have our screen or wall. With the help of some trigonometry, we'll find two triangles containing exactly the same angle α. On the one hand, sin α can be expressed by "λ divided by g" for the first secondary maximum. On the other hand we understand that sin α is given by d, with d defined as the distance to the first secondary maximum divided by L, L being the distance to the wall. So what we get here is that g is inversely proportional to d. That means, when stretching the silk stockings so that distances between pores g grow, the distance d to the first secondary maximum decreases.

M: The calculation is correct if the model assumption that there's exactly one wave emerging from each pore in the silk stockings proves to be true.

S: In actual fact, real silk stockings do not fit this idealized theory.

M: So?

S: But the concept of superposing waves is a principle that even some more elaborated theories of interference are based on.

1 e. The Mach Cone

M: The duck overtakes the water wave, just as a jet-plane overtakes a sound wave. It's the same principle: The source of the wave moves faster than the propagation speed of the wave itself.

S: Let's assume a spherical wave originated from this point. After a time t, the wave's radius is given by c times t. The duck kept on swimming to the left at a velocity v. Here we have the so-called Mach cone. The Mach cone has an apex angle α, with sin α given by c divided by v. As sin α is always smaller than 1, this solution can only exist for v being larger than c.

M: … that is, if the duck moves faster than the water wave, or the jet-plane is faster than the speed of sound. Actually, you can determine the duck's speed, given that you know the propagation speed of the aquatic waves, by measuring the angle α. The same goes for the jet-plane.

1 f. Special Relativity

S: Before Einstein, there existed actually no precise understanding of what a distance really is in the real world. The only thing that can be computed in Euclidean geometry are distances in space, or rather, merely distances in flat space.

M: This formula for computing distances is based on Pythagoras' theorem, not more, not less.

S: That's right. The distance from the point x to the origin of coordinates is given by the square root of the sum (x-i) ^2 with x-j being the j-th spatial coordinate.

M: However, we understand we don't live in a mere space, but in space-time. Let's give a simple example: When I make a full turn around myself, I arrive at the same point in space, but not at the same point in time, as time proceeds. So what kind of distance can we measure in space-time?

S: Einstein was the one to determine the correct formula. The spatial proportion is the same as with Euclid. The time contribution is dt^2. These two elements are subtracted from each other. For the light goes ds^2=0.

M: Let's introduce another inertial system, moving with velocity v relative to the first one. Then, space and time coordinates expand and compress, only the speed of light c remains unchanged.

S: There's one very special inertial system: the so-called *Eigensystem*. In the *Eigensystem*, I literally lack any distance to myself, so that the special distance dx becomes zero. The only thing that proceeds is my proper time d-tau.

M: For the light d tau is zero, so from the point of view of light there is no such thing as time!

S: In some sense, the concept of time is applicable to massive objects only. Let's continue to discuss massive objects. An outside observer moving at speed v would measure a different time dt, which longer than my *proper time*.

M: Such a dilatation of time was first observed in muons stemming from cosmic radiation, a finding which confirmed Einstein's theory.

S: Next we'll introduce a quadruple vector, with time on the first and the three spatial coordinates at the bottom. Now, we repeat the transformation into the Eigensystem. In the Eigensystem, the spatial coordinate is zero. Next, we'll subject this quadruple vector to an interesting mathematical operation: We take the derivative with respect to the proper time. We'll be doing that both in a system moving relative to the Eigensystem and in the Eigensystem. In my Eigensystem, we get a constant (the speed of light) on the time coordinate, and of course zero in the space coordinates. But what is this new quadruple vector in a system moving relative to myself with velocity v? Multiplying both sides of the equation by the constant "rest mass m0 multiplied by c", we obtain an energy in the time component. The derivative of the position with respect to proper time is the relativistic generalization of the momentum p, multiplied by c. So we can write down Energy on the upper left side, at the bottom momentum p times speed of light c. What we get is an interesting new formula waiting for an interpretation. When I rest in my Eigensystem, I still have an rest energy, given by the rest mass m0 multiplied by c2.

M: An observer who is moving at a speed v relative to me measures my energy as E and my momentum as p. These quantities can be put in relation by writing down the invariants.

S: Just as a few moments ago we discovered the relation of time and spatial coordinates in a moving system to the invariant "proper time" …

M: …we'll now notice the relation of energy and momentum of the moving particle to the invariant "rest energy".

S: Equivalent to this formula is Einstein's famous $E = m c^2$, whereas here the mass is not the invariant rest mass m0, but depends on the velocity – For higher velocities, the mass increases…

M: For training purposes, it might be a nice exercise to derive $E = m c^2$ from this formula.

2 a. The photoelectric effect

M: Next up: the photoelectric effect The glass pane filters out blue and ultraviolet radiation. As long as that happens, the pane remains negatively charged, which in the figure is indicated by minus signs.

S: In our model, the electrons are frogs in a prison yard, who must surpass the escape energy W to make it out of the prison yard, or rather, their potential well**.** After beaming in the light energy h ν, the following applies: the surplus energy h ν - W, is the kinetic energy of the electron escaping from the metal plate.

M: Observe the following experiment: We'll be beaming in light of different frequencies ν, and measure the kinetic energy of the escaping electrons – this can be achieved, for example, by applying a countervoltage. According to the equation h ν - W we get a straight line; with the slope being Planck's constant h. One of many many possibilities to measure the value of Planck's constant.

2 b. Planck's Formula

M: This light's colour tells us how hot the wire is. We're able to calculate the temperature using Planck's radiation formula. The essential assumption here is that there are energy transitions inside the wire which are capable of absorbing and emitting photons of any given frequency – this defines the "black body".

S: For deriving Planck's radiation formula, we should first examine one two-level system, where the electron in the ground state goes to the excited state when absorbing a photon of matching energy – in the figure, the photon is indicated by a wavy line.

M: During the reverse process, the electron falls down, out of the excited state into the ground state, emitting a photon.

S: Up to this point everything has been easy – but there's a third mechanism, which was first suggested by Einstein. It's the so-called "induced emission". A photon with the matching energy stimulates the electron in the excited state to fall down into the ground state, emitting another photon in the process.

M: With his understanding of induced emission; Einstein mathematically predicted that one day man would be able to engineer laser devices.

S: Good point - but let's get back to the radiation formula. We'll now formulate a rate equation. Because the electron is able to assume only one of either two positions, viz. ground state or exited state, in equilibrium all terms where the electron is in the ground state must be equal to all terms where the electron is in the excited state. Now, we introduce ng as the number of electrons in the ground state, and ne as the number of electrons in the excited state. u(ν, T) is the energy density of photons with frequency ν at temperature T. A and B are constants. The constant A for the two processes involving light acting on the atom is the same.

M: What's still missing to complete the computation of the rate equation is some piece of information from thermodynamics, which will simply be used without prior derivation. The ratio of atoms in the excited state to those in the ground state is in the thermal equilibrium given by the Boltzmann factor, with k T being the medium thermal energy per degree of freedom.

S: After inserting this into the rate equation, we're in a position to describe the energy density of photons with frequency ν at temperature T as follows: The so-called Bose-Einstein factor describes the photon statistics. It is the expectation value for the number of photons with a fixed wave length corresponding to the energy amount Delta E.

M: Only thing left to do is to determine the constant B through A …

S: ... … for which we need a phase space argument: We're summing up all photons of the wave length lambda that correspond to the energy Delta E, after that we'll multiply by two taking into account the two polarization directions. Finally, we multiply by h ν, since we're computing an energy density. What we get then is Planck's radiation formula.

M: In this formula, Planck's constant h occurred for the first time in history of physics as an experimentally determined constant. The significance of the constant was unknown at that time. Planck did not derive this formula presupposing quantization of radiation, but by means of an argument that examines the entropy of light waves.

S: Einstein was the first to provide the right interpretation of these things by discovering the quantization of light waves in individual energy packets.

M: Which is not to belittle Planck's achievement, not the least bit!

3 a. Vector fields

M: Now, we'll examine vector fields - to be more specific, vector fields that may originate from a mountain landscape like this one here.

S: We're able to describe mathematically the mountain landscape by 'phi of x and y', while we obtain the contour lines by searching for the solutions to the equation 'phi of x and y = constant' in the figure.

M: Those solutions for different contour lines are marked here. At right angles to these contour lines runs the gradient field from the peaks down into the valleys. The direction goes from top to bottom. That way the mountain creates the gradient field, which we're drawing in here. Peaks are linked to valleys, there's no direct connection from minus to minus or plus to plus, only from plus to minus. At right angles to this gradient field the curl field is located. This vortex line winds its way around two peaks.

S: Next we'll try to mathematically describe these two kinds of fields. The gradient field in three dimensions has the following essential properties: A gradient field's rotation is zero, the divergence is not.

M: That means the gradient field has a starting point and an end point, has sources and sinks and runs from peak to valley. With the curl field it's vice versa. A curl field's rotation is not zero, the divergence is zero. That means: the curls always run in circles, they do not start or end anywhere.

S: I should add that not all curl fields are orthogonal to a gradient field. When the space itself is not flat, topological curl fields may exist. For example, the section of a sphere with a mountain landscape automatically creates a curl field.

3 b. Electro- & Magnetostatic

M: We draw the vortex lines into this magnetic curl field. The magnetic field is called B. The statement that there are no magnetic charges is equivalent to: The divergence of B is equal to zero. That means: no sources and sinks. The rotation of B, however, is unequal to zero. In contrast, it is possible for the electric field E to begin and end at charges, in this case, for example, the right side is positively charged and the left side is negatively charged. The divergence of E is unequal to zero. At this point we will not discuss the rotation of E yet. Only for static electric fields it is zero, not for dynamic electric fields.

S: The most basic model for an electric field is the point charge, its potential being 'e divided by r'. Its electric field is obtained as the negative gradient of this potential. Working this out, one obtains {e divided by r squared}, with the unit vector pointing in the r-direction.

M: … here, e is the electron's charge.

S: The divergence of E is equal to minus Laplace of the potential, i.e.: minus Laplace e divided by r, which is equal to 4 pi e times the Delta function in the point x, and the point x=0 is where the said charge is located at. That's where the gradient field's electric flux lead to.

M: For the following considerations we need some additional charges. So we're inserting some additional valleys and peaks into our potential landscape, subsequently connecting them to gradient fields, linking plus to minus in the process, never minus to minus.

S: In mathematical terms, we're adding up in Delta functions; a charge is found in each point x-i, the charges are given the index i, because they may be positive or negative. In general, the divergence of B is equal to zero and the divergence of E is equal to the charge density rho. The electric field starts and ends at charges …

M: ... however, magnetic charges ("monopoles") have not been observed so far. The magnetic field does start or end at charges, but always runs in circles. Now, you may wonder, what is the situation like with electric fields? Perhaps electric fields might rotate, too ? Is there something like electric curl fields?

S: Well, not in electrostatics. But what about electrodynamics, what's the situation like there?

3 c. Electrodynamics

M: Electrical charges create an electric gradient field. Magnetic charges have never been measured. Consequently, magnetic gradient fields do not exist.
But is there something like electric curl fields? Well, in electrodynamics electric charges are allowed to move, and we understand from experiments that moving electrons generate a magnetic curl field. Magnetic charges, however, do not exist. So it's impossible to create an electric curl field by means of a stream of magnetic charges.

S: But we still have an ace up our sleeve: We have not looked at the phenomenon of *time-variant* electric and magnetic fields yet.

M: And indeed, by way of experiment we identify a time-variant magnetic field produces an electric curl field. This is the so-called law of induction which e.g. the functioning of a dynamo is based on. The minus sign indicates the fact that electric curl field ‘oppose’ the temporal changes of the magnetic field. That is the so-called Lenz’s rule.

S: A time-variant electric field, in turn, creates a magnetic field. This is Maxwell’s so-called displacement current – it’s the only thing Maxwell contributed to these equations! As time-variant electric and magnetic fields may generate each other and thus interplay with each other, this complete set of equations permits for electromagnetic waves as new possible solution. It was Maxwell’s achievement to see this.

M: We should put all this into practice in an experiment. Let U0 be an alternating voltage producing an alternating current flowing through this thick white coil consisting of some 15 turns, running back and forth and back and forth etc. This alternating electric current generates a magnetic curl field. The magnetic curl field varies in time, because we are applying an alternating voltage. The time-variant magnetic curl field generates an electric curl field orthogonal to the magnetic curl field: rotation E is proportional to minus B dot.
Inside this big copper coil the electrons are subject to a force in this electric field, which goes for every single one of the many turns in the coil. The higher the number of turns in the coil, the higher the number of electrons affected by the electric force. So…what we have built here is a voltage transformer. Assuming the big copper coil has N1 turns and the little white coil N2 turns, the induced voltage, which may be measured at the top of the coil, is given by the N1 divided by N2 multiplied by the input voltage U0. There's no direct connection between both coils. The force is transmitted by the electromagnetic field through the space . This works only with alternating current, as only time-variant magnetic fields are able to generate an electric curl field.

3 d. Electric and Magnetic Forces

M: We notice electric and magnetic fields only because they exert a *force* on charges. A very powerful electric field is located near this metal point, exerting the force ‘q times E’ on airborne charges q. Due to the electric field’s enormous size the atoms in the air are ionized, with the charges being torn away from the atomic nucleus. That’s what generates these lightnings.

S: Once again we go back to the most basic scenario of a single charge Q generating an electric field in space – we’ve already computed this. A test charge q in this field experiences a force ‘q times E’, i.e. q times Q divided by r squared in the direction of the vector connecting these two charges. Magnetic fields only act on moving electrons - that is the so-called Lorentz force. The Lorentz force depends on the direction in which the electrons are moving. The Lorentz force is located at right angle both to the magnetic field and to the direction of the electrons’ motion and thus is proportional to ‘v cross B’.

M: In the laboratory, we observe moving electrons being affected by a magnetic field. However, if we flying together with electron, the speed v will be equal to zero, and thus there is no Lorentz force. But the electron still senses a force. This can only be an electric force, which leads us to the conclusion that the field which is magnetic in laboratory is from the electron’s perspective an electric field E’.

S: … i.e. E’ equal to v divided by c cross B’.

M: … with v being the relative velocity between the two inertial systems.

3 e. Maxwell’s equations

S: Let’s have a look at the complete set of the Maxwell equations.

M: Alright, let’s begin with the experimental fact that while electric charges do exist there is no such thing as a magnetic charge.

S: Stationary electric charges generate an electric gradient field. Moving electric charges generate a magnetic curl field. The divergence of E is equal to 4 pi rho. The rotation of B is equal to 4 pi/c times j, with rho being the charge density and j being the current density. From the fact that magnetic charges do not exist follows that the divergence of B is equal to zero and the rotation of E is equal to zero for stationary fields. Next we'll examine time-variant electric and magnetic fields.

M: The so-called law of induction was first discovered by way of experiment. A time-variant magnetic field generates an electric curl field.

S: By integrating the corresponding equation over one area, we obtain the time derivative of the magnetic flux. It generates an induction voltage U ind. Next, we discuss Maxwell's displacement current in the last equation. It is only by adding that term that the equations become complete, permitting to have electromagnetic waves as a new solution class.

M: These equations wouldn't make sense if the electric and magnetic fields didn't show up through forces they exert on test charges q. The electric force acting on the charge q is 'q times E', the magnetic force – or Lorentz force – is 'v divided by c cross B'.

S: As a most basic example, think of the point charge Q generating an electric potential whose negative gradient is the electric field. A second charge (q lower case) placed somewhere inside this potential senses the force q times E. This formulation of Maxwell's equation is asymmetrical. On the one hand, there are charges *generating* electric and magnetic fields. On the other hand, there are test charges *sensing a force* in these fields. The electric field always acts on other charges only, the "self-interaction", or force exerted by the field on the charge itself, is ignored.

M: If you'd naively evaluate the field that generates the charge Capital Q at the very position of the charge itself, setting r = 0, a terrible divergence would be created. In Maxwell's time this problem was simply ignored. The field is supposed to act only on other charges, not on the charge creating the field. It is this separation that causes the formulation of the equations to become asymmetrical.

S: In quantum electrodynamics, the asymmetrical treatment of the field is improved. However, the mathematical problems posed in the discourse on occurring divergences are not too easy. But the theory's success is indeed very convincing.

Next, we discuss the potential of a point charge in various dimensions. We'll start with three dimensions – the potential being given by 1 through the distance to the point charge. Now, we carry out the Fourier transformation into momentum space. Let k be the wave vector. Obviously, the factor 1/k^2 needs to be integrated over all the momentum modes to obtain 1/r as the potential in local space. In D dimensions, solving the equation 'Laplace V equal to minus Delta function' is very easy in momentum space: Instead of three k integrals we're writing D integrals. As a result we simply obtain - for D unequal to 2 – as result 1/r^(D – 2).

M: And what's the situation like with 2 dimensions?

S: In D = 2 dimensions, the potential behaves logarithmically.

M: This is the potential created by an infinitely long, charged wire.

S: Very important are 4 dimensions, as we're able to rotate time as the fourth dimension formally in the complex plane, x4 = i t. Within this "Euclidean" formalism the photon propagator is simply given by 1/x^2.

4 a. The atom and the solar system

S: Newton succeeded in developing a theory of gravitation allowing for the planets' movements around the sun to be described mathematically. Gravitational force and electric force are very similar.

M: Both are proportional to the squared distance between the two charges.

S: Then why not describe the electrons revolving around in the atomic nucleus of the hydrogen atom in analogy to earth revolving around the sun?

M: In both cases Newton's classical solutions of the equations of motion are conic sections; for the bound system just ellipsoids.

S: For the electron in the atom, that assumption is wrong. Experiments tell us that the spectrum of emitted radiation of the atom is *discrete*. This can by no means be explained by electrons revolving around the atomic core in a *continuous* fan elliptic trajectories. Neither is Bohr's approach considering only a discrete set of orbits a sustainable concept. What's needed here is a new, a revolutionary idea.

4 b. The commutator

M: Let's start the search for this revolutionary idea with - this glass of water. What can we do with that water glass? You can turn the glass upside down. By turning it around again, you'd restore the initial state. You can pour in water. Pouring more water into the full glass won't change the state of the glass, as it remains full. After turning it upside down, it's empty and upside down – nothing to it!

S: Taking things to a more abstract level, we'll introduce two operators. The U-operator, which turns the glass upside down and the W-operator, which pours out the water. What happens when applying U twice? Turn once, turn twice, which brings us back to where we started, back to the initial state. When applying W twice, however, we get a full glass, and after that, a full glass again. So U U applied to 'glass up' is different from W W applied to 'glass up'. Next we'll try applying a combination of U and W on 'glass up', for which applies: U W on glass up is equal to W U on glass up. However, things are different when starting with glass down as an initial state.

M: Who wouldn't turn the glass from upside down into its regular position *before* pouring in water – instead of doing it the other way round!

S: For the operators U and W that means it is important to consider the order of application on the state of the object. The operations don't commutate. W U applied to glass down is not equal to U W applied to glass down. Next, we introduce the *spectral representation* of the operators:

M: To do this, we first have to define the basis set of all possible states of the glass. This isn't hard to tell. Glass up - empty, glass down - empty, glass up - filled…

S: That defines the basis for what we're going to do, and we define it to be an orthonormal basis. Let the scalar product between two identical states be 1 and all other combinations zero.

M: Is that a definition?

S: Exactly. The matrix formed by all possible combinations of two basic vectors has nine entries – here's the unit matrix. In the case of our inversion operator, the matrix looks like this: We write "1" in the place where the conditions are identical. Our example shows the spectral representation of U in the basis defined above.

M: In the same manner, we can obtain a spectral representation of W.

S: We have found two matrixes representations for the operators. This way, the two operators have been given a mathematical shape. Now we're in a position to compute the commutator between the two operators by means of matrix multiplication: U W – W U, the result being another matrix, which is unequal to zero. Thus, the commutator is unequal to zero.

M: These considerations have yielded the following results: If there are operations that do not commutate with each other, then these can be mathematically described by means of matrixes. (When the number of basis vectors is finite) The operators act on states. The states themselves are in general are no operators – they are modified by operators. The number of all possible states yields the dimension of the matrices in the spectral representation.

4 c. The Schrödinger equation for the hydrogen atom

M: Newton's mechanics fails when it comes to describing the atom, because it was proven by way of experiment that atoms are capable of emitting only a discrete radiation spectrum, so that the electron's permitted energy levels in the atom must be a discrete staircase. The classical Hamiltonian H describes the energy of a classical point-like electron. Newton's classic equation of motion is the solution of the differential equation „Force equals mass times acceleration". In classical physics, any kind of elliptic trajectory and thus any energy is allowed.

S: A crucial thought turns classical mechanics into quantum mechanics. The commutator of position and momentum is unequal to zero. Postulating the commutator of position and momentum as i h/(2 pi) yields *mathematically* an energy staircase corresponding very accurately to the observed energy staircase.

M: This fantastic result revolutionized the theory of atoms - nature makes us accept that position and momentum are operators. Although the commutator is very small, - h is a very small action – it is indeed unequal to zero.

S: That makes all classical functions of location and momentum become operators – we're marking these with a red hat, while i and j are the spatial coordinates. With no ordinary functions left, stuck with nothing but operators, these are supposed to have an effect on states. But what are the states?

M: In the hydrogen problem, the state corresponds to the electron being located on a particular step on the energy staircase. Each step on the staircase carries the natural numbers (n, l m), with (n) being the so-called principal quantum number and (l, m) the angular momentum quantum numbers.

S: Interpreting the mathematical expression for the state (n, l, m) in the point x, the so-called wave function of the electron in the point x, is not an easy task to accomplish. The formal computation, in turn, is astoundingly successful: The so-called stationary Schrödinger equation is a differential equation mathematically describing the energy staircase. The energies depend on n only and are proportional to (-1/n^2). The lowest energy is n=1, the next higher one is n = 2 etc. The light quantum emitted upon the electron falling down from the n=2 to the n=1 state corresponds exactly to the energy difference Delta E = E2 - E1 =h ν

4 d. The angular momentum algebra

M: On with angular momentum algebra. All we need for that is a staircase having a particular number of steps and a ball placed somewhere on it. The ball can leap up the stairs -

S: …and down the stairs.

M: And it can stay where it's at. That's all. The most basic staircase includes only two steps. In that case, there are only two options for the ball: It may either be located at the top or at the bottom.

S: If the ball is to leap upstairs then that implies for us to perform an operation – thus we'll introduce an operator taking the ball one step up, the so-called ascending operator J+.

M: So what happens if we apply J+ for a second time?

S: When re-applying J+, the ball disappears from the stairs. J+ applied to ball at the very top yields zero. What we have here is the spectral representation of the ascending operator in the basis of the ball's position on all possible steps.

M: In analogy to that, there's the descending operator whose only purpose is to take the ball one step down. The spectral representation of that is once again a 2 cross 2 matrix, or simply the transposed of the spectral representation of J+.

S: Then there's a third operator, which tells us where the ball is at. To explain this, we'll need to give *names* to the individual steps. The names we'll choose in such a way that the height of the individual step be one and the names be symmetrical around zero.

M: These conditions unambiguously define the steps' names, so that we get: (-½) and (+½) for the two-step stairs.

S: That completes the spectral representation of the angular momentum operators for staircases with two steps.

M: These three matrixes enable us to compute the commutators.

S: That's right. The commutator of J+ and J- yields twice J z. Those who won't believe it may want to perform the matrix multiplication (J+ J-) - (J- J+) and the result is indeed 2 Jz. The same thing can be done with the commutator J+/J- and Jz which provides us with all three possible commutators that can be formed with these operators. Together, these three commutators are the so-called angular momentum algebra.

M: By the way, it's considered the most important algebra in all physics as it determines the essential characteristics of the periodic table of elements.

S: We'll do the whole thing again, this time involving a three-step staircase. Again, the steps' names are supposed to be symmetrical around zero; the step height to be 1. That unambiguously defines the names of the three steps. What does the spectral decomposition of the operator J+ look like now? It appears to be a 3 cross 3 matrix.

M: Why does it say square root of two there?

S: We normalize the operator in such a way that it fulfills the angular momentum algebra which we introduced a few moments ago, hence the normalization. J- again is the transposed of J+, Jz again is the name operator. On the diagonal, we have the names of the three steps: –1, 0, +1. The staircases' names appear to be half-integral for any even number of steps and integer for any odd number of steps. Indeed these 3 cross 3 matrixes fulfill the angular momentum algebra. So the angular momentum algebra has different representations – depending on the number of steps. We're now going to proceed from algebra to the Lie group. That is achieved by applying the exponential function to the operators from the angular momentum algebra. The two-step stairs yield the SU(2) group, while the three-step stairs yield the SO(3) group.

M: Basically, SO(3) is the group of all rotations in real, three-dimensional space.

S: SU(2) is the group describing the rotations in a space of half-integer steps. The following observation should be exciting: In real space, I need to turn around 2 pi or 360 degrees to make a full turn. In the SU(2) group corresponding to the stairs with half-integral steps, I need to turn around 4 pi, or 720 degrees, i.e. twice as much, for a full turn.

M: We can simply deduce this from the fact that 4 pi times step ½ is equal to pi times step 1. In terms of physics, the two-step staircase corresponds to the electron spin with the spin being -1/2 and +1/2. People sometimes have problems to imagine the properties of spin because of this strange double-rotation property.

4 e. The angular momentum quantum numbers

S: Next, we discuss the general representation of the angular momentum algebra.

M: The number of steps is either even or odd.

S: And the difference between the two is tremendous! The odd-numbered staircase has a step # 0, the most basic staircase is a three-step one, it is a staircase of the kind which we just examined. In general, stairs may have (2 l+1) steps, with the nomenclature being defined by the following properties: (i) Step height being equal to 1 (ii) Steps' names symmetric around zero. The representation of the operators is thus given by 2l+1 times 2l+1 matrixes.

M: In physical terms, that corresponds to angular momentum, i.e. the angular momentum in real, three-dimensional space.

S: If the ball is located on some step m, then obviously that implies that it is located somewhere between –l and +l.

M: m is the so-called “magnetic quantum number”

S: For any even number of steps, zero is not a possible step name and the steps’ names are half-integral. In general, there may thus be steps # ½. # 3/2 and so on up to step # S, and in a symmetrical manner down to step # -S. There are 2 S steps, with S being half-integer.

M: In physical terms, that corresponds to the spin S. For that spin, the starting point is not reached until a turn by 4 pi has been made.

S: And again the ball is located at some point s somewhere on the stairs.

M: Returning to the Hamilton operator with the commutator px – xp, all we need to do is to scale the steps. The steps in angular momentum algebra all have the same height, yet their height is not 1, but given by the Planck’s constant h.

S: ... which can be shown by carrying out a short computation involving the commutator of position and momentum.

M: At an early stage in the development of quantum mechanics, only staircases with and odd number of step had been discovered (representing the quantum mechanics counterpart of angular momentum in real space). The two quantum numbers l and m are nothing else than the house numbers m on a staircase having an odd number of 2 l + 1 steps, like the one we just saw.

S: The quantum mechanical computation – the solution to the differential equation for the electron’s wave function – tells us l may only assume values between zero and n – 1, for fixed principal quantum number n. This result is a very important one, which we cite here without proof.

M: If the principal quantum number n is one, l must be equal to zero. At the early stage of quantum mechanics, the electron spin s=1/2 hadn’t been experimentally discovered yet.

4 f. The groups SU(2) and SO(3)

4 f. I.

S: Next up: a close look at the rotation group SO(3). What does it do? We have two coordinate systems which are rotated against each other without dislocating/shifting the origin of coordinates. Let e_1, e_2, e_3 be the orthonormal basis of the first coordinate system, and $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ be the orthonormal basis of the coordinate system rotated relatively to the first one. Each base element is a three-component vector. All three vectors brought together yield a 3 cross 3 matrix M, same in the other system. The following applies: M transposed times M = unit matrix, like in the other system. That’s the defined property found in all matrixes in the rotation group SO(3). Any given rotation between the two coordinate systems corresponds to a matrix from the rotation group SO(3). We’ll now proceed to further examine the rotation group. For that we should analyze the rotation of a vector x based on a rotational matrix M, which is the mapping in R3 of x onto M times x. Indeed, the rotated vector’s length remains unchanged. That is the reason why the SO(3) is called rotation group. That means, M·x is located on the surface of the same ball that the point x is on. So is the mapping of R3 onto R3, x onto y=M·x unambiguous? Is there exactly one matrix M matching the /corresponding to the rotation of x to y? Put differently, is there a H from SO(3) for which H x is equal to x ? We can answer that question in a graphic manner. We understand the matrices M correspond to rotations of two coordinate systems against each other. For rotations H around the axis x, the equation H x = x obviously has to be fulfilled. Furthermore, as H has one freedom, it must stem from the subgroup SO(2) out of the complete rotation group SO(3). So we’ve found the answer: There is a complete subgroup fulfilling the equation H x=x, and that is the one spinning around the rotational axis x. Hence, the mapping of x onto y, y=M·x isn’t unambiguous. For any M fulfilling the equation there’s a so-called fiber M H fulfilling the same equation. Next we're going to examine a number of points y_i which are all created by similar rotations, with $y_i= M_i$ x. For any Matrix M_i there’s a fiber M_i H, and all fibers put together make for the so-called fiber bundle. Now this is where things get exciting: Even though not exactly every point y_i corresponds to a rotational matrix M_i, each point y_i corresponds to a fiber bundle $M_i \cdot H$! All points y_i together make for the ball S2. Hence, an isomorphism is supposed to be created between the ball S2 and the SO(3) group if all the matrixes created by righthand multiplication by H are identified with each other. In the quotient SO(3)/SO(2), any fiber $M_i \cdot H$ is only/represents only one point, and that point corresponds exactly to one point on the ball S2.

4 f. II

Next we'll take a look at the stereographic projection of the ball onto the plane. The point x on the ball is projected onto z in the plane, with z being equal to the length of x divided by x1. That can be generalized to three dimensions. So we take every point on the ball and map it onto the plane, by which a mapping between the ball and the complex plane is defined. However, the north pole of the ball is projected into infinity. After adding the Infinity Point to the complex plane, the stereographic projection will define an isomorphism between the ball and the complex plane + Infinity.
What does that imply for our SO(3) group ? Well, SO(3) defines a rotation on the ball of the point x to the point y. We're able to both project x onto the plane and project y onto the plane. But how to project the rotational group onto the plane? That means, which group transformation phi does the same thing on the plane that SO(3) does on the ball, which is w = phi(z), with M x = y ?
Here's the deal: The key to solving is the so-called Moebius transformation - uh, looks more complicated than it is: The 2 cross 2 matrix alpha beta gamma delta stems from the group corresponding to SO(3) after subjecting the latter to a stereographic projection onto the plane. Yet this complex matrix A is a 2 cross 2 matrix. Recalling the discussion of angular momentum algebra for a two-step staircase, we'll realize A must stem from the SU(2) group. A does indeed fulfill the defined property A dagger A = 1. So what we've found is a mapping of the "three-step staircases" group into the "two-step staircases".group. On the algebra level we saw that a full rotation, viz. one by 4 pi, in the SU(2) group corresponds to two rotations, or 2 times 2 pi, in the SO(3) group. This fact again shows here, although this time it's on group level. Because the Moebius transformations involving the matrixes A and minus A lead to the same point w = Φ(z), every rotation M from SO(3) corresponds exactly to two mappings A and –A from SU(2). That applies to every fiber, and thus to the entire group. Thus, SU(2)/Z2 is isomorphic to SO(3).

4 f. III

Next we're going to rotate the stereographic projection into complexity, for which we had to replace one of the coordinates by i times that coordinate, or x1^2 plus (i x 2)^2. That way the ball becomes a hyperboloid. The hyperboloid we're projecting/mapping onto the unit circle. In two dimensions, the point d is given by x2 through x1. In three dimensions things look as detailed in continuation: Contemplating once again a "rotation" on a hyperboloid, we demand any point that was located on it before the "rotation" be still on it afterwards. This mapping M Hut belongs to the SO(2,1) group defined in a manner very similar to the SO(3) rotational group, only that the metrics involved is not (1,1,1) but (1,1,-1). In this case again a subgroup H is displayed rotating around the point x, with H x = x and M_i H x = y_i. Thus again any point y_i on the hyperboloid is isomorphic to a fiber „$M_i \cdot H$" in the SO(2, 1) group, with all the fiber's elements created by multiplying fixed „M_i" by all the possible matrixes H from the subgroup.

4 f. IV

In continuation, we're going to map/project the hyperboloid onto the unit circle, just like we mapped the ball onto the complex plane a few moments ago. That creates a hyberbolic geometry on the unit circle. Now we're in a position to map the point z onto w = phi(z) by performing a Moebius transformation, the 2 cross 2 matrix has become an element of the SU(1,1) group defined in a manner very similar to the SU(2) group, only that the metrics involved is not (1,1) but (1,-1). Again we'd like to establish a relation between the SO(2,1) group transporting points on the hyperboloid to different points on the same hyperboloid, and the SU(1,1) group transporting points on the unit circle to different points on that unit circle. Here's once again the definition of SU(1,1) involving the metrics (1,-1). The projection of the hyperboloid onto the unit circle defines a mapping between the SO(2,1) group and SU(1,1). Just like it occurred a few moments ago, the matrixes A and minus A in SU(1, 1) correspond to one matrix in SO(2, 1).

4 g. Relativistic corrections to the Schrödinger theory

S: Apart from the concept of commutators, there's also the theory of relativity waiting to be included into our theory for the hydrogen atom. Up to this point, we have only looked at non-relativistic quantum mechanics.

M: Newton mechanics was initially improved by implementing two independent modifications: Firstly, by the commutator xp-px equal to i h/(2 pi), which yields quantum mechanics. Secondly, by the theory of relativity, which is based on the fact that the speed of light c is a constant of nature. The theory of relativity is a classical theory, it doesn't employ any operators whatsoever. Let's leave the commutator aside for a moment and remember the relativistic formula for the energy contained in a particle from the rest mass m0 with a momentum p.

M: We should compare that to the expression dating back to Newton: The rest energy m0 c^2 is a new factor here, the term p^2/2m is the kinetic energy already known to Newton.

S: … all other terms are from the order (p/c)^2 smaller and give the so-called relativistic corrections. Now we'll sort of "get the commutator to work" again, reaching quantum mechanics: We're turning x and p into operators again -

M: … and realize Newton's expression/term describing kinetic energy obviously is only an approximation ignoring the existence of the maximum speed c. Now how can both the commutator and the maximum speed c be included with the theory?

S: The most basic concept for that is the perturbation theory. In first order of the perturbative calculation, we should consider the next term from the Taylor expansion as an operator, subsequently evaluating the matrix element for the electron with quantum numbers # (n, l, m). The energy *Eigenvalues* are no longer degenerated, but depend explicitly on the angular momentum quantum numbers l and m.

4 h. The Dirac equation

S: Dirac succeeded in theoretically describing the electron taking into account both the principles of quantum mechanics and those of the theory of relativity. First, we're taking the quadruple vector of energy and momentum from the theory of relativity. We'll create quantum mechanical operators in momentum representation from it. That way, an operator is created from the relativistic relation between energy and momentum, it is the so-called box operator (second derivation to time minus the Laplace operator).

M: The most obvious thing to do, which in terms of chronology in physics history indeed was first done, is to establish a type of quantum mechanics based on that operator. However, the result for the energy staircase doesn't coincide with the hydrogen spectrum determined by way of experiment. From a present-day point of view you could say, using these box operators only works for describing integral-spin particles, i.e. staircases having an odd number of steps; that goes for spin 0 in particular. It was like a whole new world – like the discovery of semitones, the black keys on a piano – when Dirac succeeded in finding an operator whose square is the box operator.

S: Dirac indeed found that solution, and it is an absolutely unexpected one. It corresponds to *two* two-step staircases. These correctly describe (1) the electron with spin ½ and (2) provide as a new solution an additional particle, which has the same mass, the same spin ½, but opposite charge.

M: This new solution was first wrongly interpreted, only after a number of false guesses did he realize this had to be some new particle: the positron, the electron's antiparticle.

S: But in order for all these amazing interpretations of the mathematical structure to be understood, an explicit challenge had to be accomplished first. Let's write down an ansatz linear in time and space derivations, demanding that the square be the box operator. Thus, we'll realize that the γ^{μ} must fulfill the following anti-commutator: That type of algebra is the so-called Dirac algebra.

M: It took a superior mind not only to find the solution for this kind of algebra – Dirac wasn't even aware he had to use 4 times 4 matrixes – but to find the correct physical interpretation of the solution.

4 i. The Hydrogen spectrum in the Dirac theory

S: Computing the hydrogen problem with the help of the Dirac equation yields a staircase coinciding very precisely with the experimental findings. In leading order, we get the electron's rest mass m0 c^2, followed by the term proportional to 1 through n^2, which we already found in the non-relativistic Schrödinger theory. The additional corrections depend only on the quantum numbers n and j, with j being the sum of the orbital angular momentum l and the spin quantum number +/-½. α = 1/137 is the so-called fine-structure constant, which is a dimensionless number resulting from dividing a combination of the electron's charge (e^2) by the constants of nature h and c.

M: Like we said, Dirac's energy staircase depends only on n and j, and coincides most accurately with the findings from the experiment. The lowest energy level has the quantum numbers n=1 and j=1/2. In spectroscopy, this ground state is called 1 S ½. For n=2 we have the following options: l=0, j = ½, that's the state 2 S ½. l=1,j = ½ or j = 3/2, these are the states 2 P ½ and 2 P 3/2 .

4 j. The Lamb Shift

M: According to Dirac's theory the energy depends only on n and j, thus the state 2 S ½ and the state 2 P ½ contain exactly the same amount of energy, because both the main quantum number n=2 and j = ½ coincide with each other in both states. Even though the trajectory angular momentum l differs, j is identical as 0 + ½ is equal to 1 minus ½. It is exactly in these two energy levels where Lamb detected minute experimental deviations from Dirac's theory in 1947. So Dirac's theory, being in its time a scientific sensation, had to be re-examined. The minute shift between the energy levels was awaiting its theoretical explanation.

S: These findings forced the theory to continue on the chosen path, that is, proceed from relativistic quantum mechanics to quantum field theory – to 'QED'. Being theoreticians, we wonder where Dirac went wrong or what he failed to notice. The key here is to examine electron interaction more thoroughly. We'll begin that examination on a general scale, looking at the static potential 1/x in three dimensions. In the momentum space we're interpreting the "propagator" 1 through k^2 as follows: The interaction between two charged particles is achieved by the constant exchange of virtual photons. Integrating over all virtual photons with any given wave number k, we get the 1/x potential in the location space.

M: Up to this point, nothing but new bottles for old wine.

S: Hold it! Here's what that new idea in quantum electrodynamics looks like: This static potential is created by the most basic exchange process of virtual photons. We should discuss the question if these virtual processes might be more complicated after all. In a vacuum environment, the virtual photon might decompose into a virtual pair of electron and positron, and then become a virtual photon again. This is the so-called vacuum polarization and it causes a correction of the Coulomb potential to take effect. The second correction is about the electron's interaction with its own field, which we didn't take into account covering Maxwell's theory.

M: That's right, the field generating a charge affects only other test charges, never does it affect the generator's charge itself. Hence, the formulation established by Maxwell was asymmetrical.

S: QED can do better than that! The so-called self energy in the electron cannot to be neglected: The electron emits a virtual photon, and then absorbs it again.

M: The problem of occurring divergences is difficult.

S: Right, but someone found a solution to it. Then there's the so-called vertex correction; which with a view to what we just learned should be almost self-explanatory. The concrete computation of these corrections is indeed very complicated, mainly due to the occurring ultraviolet and infrared divergences. Following the legendary 1947 Shelter Island Conference, where the leading figures involved with the development of that theory had discussed the problem, these computations were refined step by step.

M: ... and compared to the experiment.

S: These three types of corrections – on the so-called 1-loop level – largely correspond to the experimental findings, with deviations not exceeding fractions of a percent. Examining increasingly complicated exchange processes involving virtual particles, a theoretical value is obtained that coincides astonishingly well with the experimental findings also undergoing constant refinement. This is what the esteem/reputation of QED is based on, which has made a major impact on modern theoretical physics. It became the blueprint for all other field theories that followed QED.

M: These figures are from two entirely different worlds, the experimental value originating from some most complex spectroscopic measurements using equipment the development of which Lamb et al. had driven forward due to the important role of radar in WWII, providing the technical prerequisites for the 1950s measurements in the process.

S: The theoretical value comes from the world of mathematics, of complicated divergent integrals, of examinations of symmetries and invariants – it's like two musical instruments trying to tune in to one another, trying to strike the right note, which originates from the depths of nature, and even being successful at it.

M: Yet, both tones are searching in the dark. They're only shadows of the reality that lies hidden in nature.

5 a. Light-Light interaction

M: Light interacts with light through virtual photons. The corresponding experiment, which here is re-conducted for your enjoyment, will be described in continuation: Polarized light interacts in the quantum vacuum with a constant magnetic field. That interaction causes the polarization direction to be inversed. This low power experiment is one of the few testing a still unconfirmed theoretical prediction of quantum electrodynamics. The reason for that is clear: Creating a genuine vacuum is very difficult, since countless interferences result in a minimal rotation of the polarization angle, which requires it to be regarded separately from the actual QED-effect itself.

S: Within the theory, it's much less complicated. As we're simply contemplating the QED-Lagrange function, by definition we're bound to obtain pure QED effects. Light doesn't interact directly with light, but only indirectly through virtual electron-positron pairs. Integrating out the fermions from the theory, we obtain a low energy theory for the photons called nonlinear electromagnetism or Euler-Heisenberg Lagrange function. By means of the Lagrange function, the rotation of the polarization angle can be computed.

5 b. Scattering experiments with electrons and positrons

M: In high energy physics, electrons and positrons are accelerated to very high energies, scattering each other in a particle detector. The scattered particles, or new ones being created in the reaction process, spurt out of the reaction center into particular space angle sections of the detector, where they are registered. Among the measurable and computable data in this are the so-called scattering cross sections, i.e.: How many particles of a given type will arrive after the scattering in a particular space angle element d omega? First, we'll examine the electron-electron scattering. Two electrons in, two electrons out – by 1,2 we denote the entering ones and by 3,4 the exiting ones.

S: In the leading order, there are only two matrix elements, which are also called Feynman diagram. They are created by exchanging 3 with 4. The scattering cross section is proportional to the squared absolute value of these two contributions. Each one on its own would yield the Rutherford scattering cross section. However, in the addition of the complex matrix elements, an interference term is created...

M: ... and all that even coincides with the experimental findings.

S: The reaction we're going to cover next is even more exciting: An electron scatters with a positron. Here, 3 and 4 are not interchangeable, because the two particles are distinguishable from one another. However, a second contribution is added to that, due to the two particles initially transforming into light and then back into an electron-positron pair again.

M: Let's turn to high energies now! Here, an interesting question comes up: What is the maximum length scale up to which the theoretical predictions of QED exhibit fair accuracy in describing the measured data, beyond which scale do they fail, requiring them to be supported by other means? When will new physics enter the stage?
In the early 1980s W and Z bosons, carriers of weak interaction, were discovered at CERN

S: Thus, the theory of quantum electrodynamics represents only the first piece of a puzzle in the description of the different kinds of interactions, which culminates in the so-called standard model of high-energy physics.

5 c. Feynman-diagrams

S: Yes that was a somewhat kittenish kind of representing the computation of the anomalous magnetic momentum by means of Feynman diagrams, as in the lowest order only one of these is needed. '1' is written in square brackets – it is the value predicted by Dirac's theory. All other corrections result from QED. Schwinger was the first to compute the corrections to the anomalous magnetic momentum, although not by means of the Feynman method, but by using his own approach, the so-called "source theory".

M: Schwinger was dissatisfied with the fact that Feynman's method was much more popular than his own contribution.

S: By the way, the one who succeeded in proving the equivalence of Feynman's and Schwinger's approaches was Dyson. What's more, he proved the renormability of QED on a general level.

5 d. Julian Schwinger: "On gauge invariance and vacuum polarization"

S: Ah, that's a paper Schwinger submitted in 1950. „On gauge invariance and vacuum polarization". This is a fundamental work on QED. Here, Schwinger first published many important results. One of these is the computation of the electron's anomalous magnetic momentum; the first radiative correction alpha/2 pi with alpha=1/137 being the fine-structure constant. Which were the key ideas for these computations? Apparently, these are gauge invariance and vacuum polarization, or polarizability of the quantum vacuum. For explaining that, we first need to get back to the Maxwell equations and rewrite these, creating a modernized version. First, we'll introduce a new kind of field, the so-called vector potential A. We define B to be a rotation of A. Doing that will automatically cause the equation "divergence B=zero" to be fulfilled. Using this A we're able to describe the electric field in a general way as the negative gradient of the scalar potential minus the time derivative of A.

M: Using the functions A and phi two of Maxwell's equations are automatically fulfilled. Furthermore, in the two remaining Maxwell equations we are able to express E and B by A and phi, thus obtaining a differential equation not for 6 field components but only for four field components.

S: We can arrange these four field components in a space-time vector with the components Aμ. Next we'll introduce the field intensity Fμν and the so-called action S[A]. Variation of the action yields the equations of motion, which are nothing else than the two remaining Maxwell equations formulated for A and Φ. So using the so-called "gauge field" Aμ and the action functional S[A] we have to formulate a new, modern-day version of Maxwell's equations.

M: This works for an electron and a positron, too.

S: We have to get back to non-relativistic quantum mechanics and the commutator once again and ask: How to describe the electron in an exterior magnetic field?

M: This works in quantum mechanics with the so-called minimum coupling: Substituting p by p- e/c times A, with A being the vector potential. Inserting that expression into the Hamilton operator, we obtain two new terms, one of which is -e/(2 m c) (p A + A p)+... . When the exterior magnetic field is constant, the term just mentioned is equivalent to –e/(2 m c) L B, with L being the angular momentum operator. That means an electron in a magnetic field senses any change in the energy levels in units of e/(2 m c), which is the so-called Bohr Magneton.

S: In the Dirac theory, the spin is added to that, so one obtains the contribution '(L + two times S) B'. And this very factor two represents the relation between [the quotient between?] the energy contained in the magnetic field of the orbital angular momentum and the spin predicted by Dirac's theory. As it was the case with some of the previous computations, the Dirac equation, too, may be obtained through effect variation, the consequence of which will be detailed in continuation: We have theoretically described both electromagnetic fields and the electron and positron – or spin ½ particle - in a modern way, and can now go ahead and simply add up the two effects. L1 is the Lagrange density for the electromagnetic field, L2 is the Lagrange density for the electron. Now how does the ½ spin particle interact with the electromagnetic field?

M: That corresponds to the relativistic generalization of the minimum coupling p to p minus e through c A to a quadruple vector.

S: Right, the generalization of the minimum coupling is based on an underlying principle, which is gauge invariance.

M: Well then?

S: For explaining that, we'll introduce a local phase transformation of the four component vector ψ. But doing that means the effect will change! The change occurs because the derivation affects the phase, creating this new term. We go ahead and generalize the idea of minimum coupling by introducing the covariant derivation Dμ. We demand that the gauge field Aμ restore the effect invariance under the local phase transformation. Therefore, the gauge field also needs to be transformed. In fact, the entire effect becomes invariant if the derivation is replaced with the covariant derivation. That's the so-called U(1) gauge transformation on ψ and A, and the so-called U(1) invariance of the effect S.

M: However, the effect has been supplied with an unambiguously defined interaction term in a wonderful way.

S: The drama brought about by this can hardly be put in words. Even if the formulas for the electromagnetic field and for the electrons/positrons look just like other stuff from ordinary quantum mechanics, they entail a completely new interpretation due to this kind of coupling. ψ is no longer the ordinary wave function it was in the hydrogen problem, but is now a field operator. The number of particles is no longer retained. Due to the coupling, ψ describes, when applied to the quantum vacuum, the superposition of the amplitudes of a one-electron-state with those of an electron + a photon, and with any number of additional virtual particles.

M: Using these approaches Schwinger managed to establish a mathematically correct description of the experimentally observed deviation of the electron's magnetic momentum from the value 'two'.

The interview part

6 a.

S: By integrating out the fermions from out of QED's Lagrange function, we obtain the effective low energy theory, the nonlinear electromagnetism with classical electric and magnetic fields, which is described by the Euler-Heisenberg Lagrange function. From the field of gravitation we've only covered the classic effective low energy theory – Einstein's theory of gravitation. So the major question is: What does the fundamental theory of quantum gravitation look like, whose fast degrees of freedom would have to be integrated out to arrive at the low energy theory again?

6 c.

M: Julian Schwinger, born 1918 in New York City, died in 1994, was given the Nobel Prize in physics together with Feynman and Tomonaga for his works on QED. A very important contribution was his computation of the first radiative corrections to the electron's anomalous magnetic momentum.

6 d.

S: Let's write down the Einstein equation first, which by the way results from a variation of the Einstein Hilbert action as equation of motion. Einstein wrote down once the only possible additional term in this equation, the so-called cosmological constant Lambda, to abandon it later on. The importance of this term has been discussed until today, especially in connection with dark matter.

6 e.

S: What we have here is a homogenous, ball-shaped mass distribution. The gravitation potential at a point x1 located inside the ball is given by the mass m1 located within circle that may be drawn around this point from the ball's center. The potential generated by the remaining mass from the outer ring compensates. Outside the ball, the potential is simply given by the total mass, divided by the distance to the ball's center. The potential is a continuous function, as it should be.